PROGRESS IN NATURAL SCIENCE Vol. 13, No. 3, March 2003

RESEARCH PAPERS

Busemann functions on $\mathfrak{R}_{RI}(m, n)$ and $\mathfrak{R}_{IV}(n)$ *

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Received June 24, 2002; revised September 29, 2002

Abstract The Busemann function plays a very important role in studying topology and geometry of a complete Riemannian manifold. In this paper, the Busemann functions on the real classical domain of the first type and the Cartan domain of the fourth type in the explicit formulas are obtained.

Keywords: **Busemann function, geodesic ray, Cartan domain of the fourth type.**

The Busemann function plays a very important role in studying topology and geometry of a complete Riemannian manifold[1~3]. Zhong[4] has obtained the Busemann function on the Cartan domain of the first type explicitly. Zhong[4] also announced that his method is suitable for the Cartan domains of the second and third types. But Zhong's method is not suitable for the Cartan domain of the fourth type. In this paper, we first obtain the Busemann function on the real classical domain of the first type, and furthermore, by using a special isometric mapping we obtain the Busemann function on the Cartan domain of the fourth type.

1 Preliminaries

In a complete Riemannian manifold, let $r(t)$ be a geodesic ray, t be the arc length parameter, then the Busemann function determined by $r(t)$ is defined as below[4]: $\beta_r(x) = \lim_{t \to +\infty}(d(x, r(t)) - t)$, where $d(\cdot, \cdot)$ is the Riemann distance, x belongs to the complete Riemannian manifold.

Let $R^{m \times n}$ denote the set of all $m \times n$ real matrices. The real classical domain of the first type is the following domain[5]:

$$\mathfrak{R}_{RI}(m, n) := \{\boldsymbol{X} \in R^{m \times n} : \boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}' > 0\}, \quad (m \leqslant n).$$

In Aut$(\mathfrak{R}_{RI}(m, n))$—the holomorphic automorphism group of $\mathfrak{R}_{RI}(m, n)$, the mapping: $\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{X}_0)(\boldsymbol{I} - \boldsymbol{X}_0'\boldsymbol{X})^{-1}\boldsymbol{D}^{-1}$ maps $\boldsymbol{X}_0$ onto $\boldsymbol{O}$, where $\boldsymbol{A}_{m \times m}$, $\boldsymbol{D}_{n \times n}$ satisfy $\boldsymbol{A}'\boldsymbol{A} = (\boldsymbol{I} - \boldsymbol{X}_0\boldsymbol{X}_0')^{-1}$, $\boldsymbol{D}'\boldsymbol{D} = (\boldsymbol{I} - \boldsymbol{X}_0'\boldsymbol{X}_0)^{-1}$.

Let $\varphi^{-1}(\boldsymbol{X}) = \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{X}_0)(\boldsymbol{I} - \boldsymbol{X}_0'\boldsymbol{X})^{-1}\boldsymbol{D}^{-1}$, then

$$\varphi(\boldsymbol{X}) = (\boldsymbol{A} + \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}_0')^{-1}(\boldsymbol{A}\boldsymbol{X}_0 + \boldsymbol{X}\boldsymbol{D}). \quad (1)$$

Obviously, $\varphi \in \mathrm{Aut}(\mathfrak{R}_{RI}(m, n))$.

Any $\boldsymbol{X} \in \mathfrak{R}_{RI}(m, n)$ can be writen into matrix polar coordinates $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are $m \times m$ and $n \times n$ real orthogonal matrices respectively,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix}_{m \times n}$$

$$(1 > \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0).$$

By Ref. [5], $\mathrm{d}s^2 = \mathrm{tr}[(\boldsymbol{I} - \boldsymbol{Y}\boldsymbol{Y}')^{-1}\mathrm{d}\boldsymbol{Y}(\boldsymbol{I} - \boldsymbol{Y}'\boldsymbol{Y})^{-1}\mathrm{d}\boldsymbol{Y}']$ is the Riemann metric of $\mathfrak{R}_{RI}(m, n)$ invariant under Aut$(\mathfrak{R}_{RI}(m, n))$. Thus, for any $\varphi \in \mathrm{Aut}(\mathfrak{R}_{RI}(m, n))$; $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \mathfrak{R}_{RI}(m, n)$, we have $d(\boldsymbol{X}_1, \boldsymbol{X}_2) = d(\varphi(\boldsymbol{X}_1), \varphi(\boldsymbol{X}_2))$. The geodesic distance between $\boldsymbol{O}$ and $\boldsymbol{\Lambda}$ is

$$d(\boldsymbol{O}, \boldsymbol{\Lambda}) = \frac{1}{2}\left\{\sum_{j=1}^{m}\left(\log\frac{1 + \lambda_j}{1 - \lambda_j}\right)^2\right\}^{\frac{1}{2}}.$$

* Supported by the National Natural Science Foundation of China (Grant No. 10071051) and Natural Science Foundation of Beijing (Grant No. 1002004)

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For any fixed $X_0 \in \mathfrak{R}_{RI}(m,n)$, the polar coordinate of X_0 is $X_0 = U_0 \Lambda V_0$, where U_0 and V_0 are $m\times m$ and $n\times n$ real orthogonal matrices respectively and Λ is an $m\times n$ matrix defined above. By Ref. [5], there is a $\varphi_0 \in \mathrm{Aut}(\mathfrak{R}_{RI}(m,n))$ such that $\varphi_0 : (O, X_0) \longmapsto (O, \Lambda)$. Let $s_0 := d(O, X_0) = d(O,\Lambda) = \frac{1}{2}\left\{\sum_{j=1}^{m}\left(\log\frac{1+\lambda_j}{1-\lambda_j}\right)^2\right\}^{\frac{1}{2}}$ and $a_j := \frac{1}{2s_0}\log\frac{1+\lambda_j}{1-\lambda_j}$, $j=1,2,\cdots,m$. It is easy to check that $\sum_{j=1}^{m} a_j^2 = 1$ and $1 \geqslant a_1 \geqslant \cdots \geqslant a_m \geqslant 0$.

The normal geodesic ray through the points O and Λ is[5]

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix}_{m\times n}. \tag{2}$$

Then the normal geodesic ray joining O to X_0 is $\varphi_0(r(s)) =: \sigma(s)$, i.e.

$$\sigma(s) = U_0 \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix} V_0.$$

Obviously, $\sigma(0) = O$, $\sigma(s_0) = U_0 \Lambda V_0 = X_0$, $d(O, \sigma(s)) = s$.

Because the Riemann metric of $\mathfrak{R}_{RI}(m,n)$ is invariant under $\mathrm{Aut}(\mathfrak{R}_{RI}(m,n))$, for any $\psi \in \mathrm{Aut}(\mathfrak{R}_{RI}(m,n))$, the normal geodesic ray joining $\psi(O)$ to $\psi(X_0)$ is $\psi(\sigma(s))$. But for any $X, Y \in \mathfrak{R}_{RI}(m,n)$, there is $\varphi_1 \in \mathrm{Aut}(\mathfrak{R}_{RI}(m,n))$ such that $\varphi_1 : (X, Y) \longmapsto (O, X_0)$. Consequently, the normal geodesic ray joining X to Y is

$$\varphi_1(\sigma(s)) = \varphi_1(\varphi_0(r(s))) =: \varphi(r(s)).$$

Therefore, if $\tau : [0,\infty) \longrightarrow \mathfrak{R}_{RI}(m,n)$ is a geodesic ray and $\tau(0) = X_0$, then $\tau(s) = \varphi(r(s))$, where $\varphi \in \mathrm{Aut}(\mathfrak{R}_{RI}(m,n))$, $\varphi(O) = X_0$. By (1)

$$\tau(s) = (A + r(s) D X_0')^{-1}(A X_0 + r(s) D). \tag{3}$$

2 Some lemmas

The proofs of the following five lemmas are similar to that in Refs. [4,6].

Lemma 1. Suppose that $\tau(s)$ is a geodesic ray as (3), where $r(s)$ is of the form (2), $a_1 > a_2 > \cdots > a_m > 0$. We can write $\tau(s)$ into polar coordinates

$$\tau(s) = U(s) \begin{pmatrix} \xi_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \xi_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \xi_m(s) & 0 & \cdots & 0 \end{pmatrix} V(s), \tag{4}$$

where $U(s)$ and $V(s)$ are $m\times m$ and $n\times n$ real orthogonal matrices respectively, $1 > \xi_1(s) \geqslant \xi_2(s) \geqslant \cdots \geqslant \xi_m(s) \geqslant 0$. Let $\xi_i(s) = \tanh b_i(s)$, $i = 1,2,\cdots,m$. Then there are constants α_i, $i=1,\cdots,m$, which only depend on X_0 and subsequence $s_j \longrightarrow \infty$ $(j\to\infty)$ such that $b_i(s) = a_i s + \alpha_i + o(1)$, $i = 1,\cdots,m$ on subsequence $\{s_j\}$.

Lemma 2. In $\mathfrak{R}_{RI}(m,n)$, two geodesic ray $\tau \sim \sigma$[4] (i.e. τ is asymptote to σ), where

$$\tau(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} a_i^2 = 1$, $1 \geqslant a_1 \geqslant \cdots \geqslant a_m \geqslant 0$.

$$\sigma(s) = U(s) \begin{pmatrix} \xi_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \xi_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \xi_m(s) & 0 & \cdots & 0 \end{pmatrix} V(s),$$

$$1 > \xi_1(s) \geqslant \xi_2(s) \geqslant \cdots \geqslant \xi_m(s) \geqslant 0,$$

where $U(s)$ and $V(s)$ are $m\times m$ and $n\times n$ real orthogonal matrices respectively.

Let $\rho_s = d(O, \sigma(s))$, $b_i(s) = \frac{1}{2\rho_s}\log\frac{1+\xi_i(s)}{1-\xi_i(s)}$. Then $\sigma(s)$ can also be denoted that

$$\sigma(s) = U(s) \begin{pmatrix} \tanh b_1(s)\rho_s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2(s)\rho_s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m(s)\rho_s & 0 & \cdots & 0 \end{pmatrix} \cdot V(s).$$

One can check that $\sum_{i=1}^{m} b_i^2(s) = 1$.

Therefore, there is a subsequence of s such that $b_i(s) \to b_i$, $(i = 1,2,\cdots,m)$; $U(s) \to U$; $V(s) \to V$, obviously $UU' = I$, $VV' = I$. Then

$$U \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_m & 0 & \cdots & 0 \end{pmatrix} V = \begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_m & 0 & \cdots & 0 \end{pmatrix}.$$

Furthermore, $a_i = b_i (i = 1, 2, \cdots, m)$.

Lemma 3. Suppose $\sigma(s) = \varphi(r(s))$ be a geodesic ray and $\sigma(0) = \boldsymbol{X}_0$, where

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} a_i^2 = 1,\ 1 > a_1 > \cdots > a_m > 0.$

$\varphi \in \mathrm{Aut}(\Re_{RI}(m, n))$ and $\varphi(\boldsymbol{O}) = \boldsymbol{X}_0$.

$$r_1(s) = \begin{pmatrix} \tanh b_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} b_i^2 = 1,\ 1 > b_1 > \cdots > b_m > 0,\ r_1(0) = \boldsymbol{O}$.

If $\sigma \sim r_1$, then $r \equiv r_1$.

Lemma 4. Suppose

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} a_i^2 = 1,\ 1 \geqslant a_1 \geqslant \cdots \geqslant a_m \geqslant 0$. Fix $\boldsymbol{X}_0 \in \Re_{RI}(m, n)$. Then the unique geodesic ray which is asymptotic to $r(s)$ and starting from $\boldsymbol{X}_0$ must have the form $\varphi(r(s))$, where $\varphi \in \mathrm{Aut}(\Re_{RI}(m, n))$ and $\varphi(\boldsymbol{O}) = \boldsymbol{X}_0$.

Lemma 5. Suppose

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} a_i^2 = 1,\ 1 > a_1 > \cdots > a_m > 0.$

If $\varphi(r(s)) \sim r(s)$, where $\varphi \in \mathrm{Aut}(\Re_{RI}(m, n))$ and $\varphi(\boldsymbol{O}) = \boldsymbol{X}_0$, then $\varphi^{-1}(r(s)) \sim r(s)$, obviously, $\varphi^{-1} \in \mathrm{Aut}(\Re_{RI}(m, n))$ and $\varphi^{-1}(\boldsymbol{X}_0) = \boldsymbol{O}$.

We can write $\sigma(s) := \varphi^{-1}(r(s))$ into

$$\sigma(s) = \boldsymbol{U}(s) \begin{pmatrix} \tanh b_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m(s) & 0 & \cdots & 0 \end{pmatrix} \cdot \boldsymbol{V}(s),$$

where $\boldsymbol{U}(s)$ and $\boldsymbol{V}(s)$ are $m \times m$ and $n \times n$ real orthogonal matrices respectively.

According to Lemma 1, there is a subsequence $s_j \to \infty$ such that $b_i(s_j) = a_i s_j + \alpha_i + o(1)$, $i = 1, 2, \cdots, m$.

Then $\alpha_k = -\frac{1}{2}\log d_{kk}(\boldsymbol{X}_0)$ and

$$\begin{pmatrix} d_{11}(\boldsymbol{X}_0) & * & \cdots & * \\ * & d_{22}(\boldsymbol{X}_0) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_{mm}(\boldsymbol{X}_0) \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{X}_1)^{-1}(\boldsymbol{I} - \boldsymbol{X}_0\boldsymbol{X}_0')(\boldsymbol{I} - \boldsymbol{X}_1')^{-1},$$

where $\boldsymbol{X}_0 = (\boldsymbol{X}_1^{(m)}, \boldsymbol{X}_2^{(n-m)})_{m \times n}$.

3 The Busemann function on $\Re_{RI}(m, n)$

Theorem 1. In $\Re_{RI}(m, n)$, for any geodesic ray through $\boldsymbol{O}$

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix},$$

$$\sum_{i=1}^{m} a_i^2 = 1,\ 1 \geqslant a_1 \geqslant \cdots \geqslant a_m \geqslant 0,$$

the Busemann function $\beta_r(\boldsymbol{X})$ determined by $r(s)$ is

$$\beta_r(\boldsymbol{X}) = -\frac{1}{2}\sum_{k=1}^{m} a_k \log d_{kk}(\boldsymbol{X}),$$

where $\boldsymbol{X} = (\boldsymbol{X}_1^{(m)}, \boldsymbol{X}_2^{(n-m)})_{m \times n} \in \Re_{RI}(m, n)$,

$$\begin{pmatrix} d_{11}(\boldsymbol{X}) & * & \cdots & * \\ * & d_{22}(\boldsymbol{X}) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_{mm}(\boldsymbol{X}) \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{X}_1)^{-1}(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}')(\boldsymbol{I} - \boldsymbol{X}_1')^{-1},$$

Proof. (i) We first treat the case $a_1 > a_2 > \cdots > a_m > 0$.

Let $\{s_k\}$ be a subsequence of $\{s\}$ such that $s_k \to +\infty$ when $k \to +\infty$.

Let β_k be the geodesic ray joining $\boldsymbol{X}$ to $r(s_k)$ and $\beta_k(0) = \boldsymbol{X}$. Then $\dot{\beta}_k(0) \longrightarrow v\ (k \to \infty)$.

Thus there is a geodesic ray σ such that $\sigma(0) = \boldsymbol{X}$, $\dot{\sigma}(0) = v$, $\sigma \sim r$. For $\sigma(s)$, there is $\varphi \in \mathrm{Aut}(\Re_{RI}(m, n))$, $\varphi(\boldsymbol{O}) = \boldsymbol{X}$ and

$$r_1(s) = \begin{pmatrix} \tanh b_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m s & 0 & \cdots & 0 \end{pmatrix},$$

$\sum_{i=1}^{m} b_i^2 = 1$, $1 \geqslant b_1 \geqslant \cdots \geqslant b_m \geqslant 0$, $r_1(0) = \boldsymbol{O}$ such that $\sigma(s) = \varphi(r_1(s))$. According to Lemma 3, one has $r_1 = r$, thus $\sigma(s) = \varphi(r(s))$. Let $\tau(s) = \varphi^{-1}(r(s))$. Because $d(r(s), \varphi(r(s))) = d(\varphi^{-1}(r(s)), r(s))$, $\tau \sim r$. Write

$$\tau(s) = \boldsymbol{U}(s)\begin{pmatrix} \tanh b_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m(s) & 0 & \cdots & 0 \end{pmatrix} \cdot \boldsymbol{V}(s),$$

By Lemma 5, $b_i(s) = a_i s + \alpha_i + o(1)$, $i = 1, 2, \cdots, m$ and $\alpha_k = -\frac{1}{2}\log d_{kk}(\boldsymbol{X})$, $i = 1, 2, \cdots, m$.

$$\begin{aligned}\beta_r(\boldsymbol{X}) &= \lim_{s\to+\infty}(d(\boldsymbol{X}, r(s)) - s) \\ &= \lim_{s\to+\infty}(d(\varphi^{-1}(\boldsymbol{X}), \varphi^{-1}(r(s))) - s) \\ &= \lim_{s\to+\infty}(d(\boldsymbol{O}, \tau(s)) - s),\end{aligned}$$

but $d(\boldsymbol{O}, \tau(s)) = \left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}}$,

$$\begin{aligned} d(\boldsymbol{O}, \tau(s)) - s &= \left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}} - s \\ &= \frac{\sum_{i=1}^{m}(a_i s + \alpha_i + o(1))^2 - s^2}{\left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}} + s} \\ &= \frac{2s\sum_{i=1}^{m} a_i\alpha_i + o(1)}{\left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}} + s}. \end{aligned}$$

Because $\left|\left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}} - s\right| = |d(\boldsymbol{O}, \tau(s)) - d(\boldsymbol{O}, r(s))| \leqslant d(r(s), \tau(s)) \leqslant$ const., $\frac{\left(\sum_{i=1}^{m} b_i^2(s)\right)^{\frac{1}{2}}}{s} \longrightarrow 1$ when certain subsequence s_j of $s \to \infty$.

Thus for certain subsequence of s, $d(\boldsymbol{X}, r(s)) - s = d(\boldsymbol{O}, \tau(s)) - s \longrightarrow \sum_{k=1}^{m} a_k\alpha_k$.

Therefore $\beta_r(\boldsymbol{X}) = \lim_{s\to+\infty}(d(\boldsymbol{X}, r(s)) - s) = \sum_{k=1}^{m} a_k\alpha_k = -\frac{1}{2}\sum_{k=1}^{m} a_k \log d_{kk}(\boldsymbol{X})$.

(ii) We turn now to the case $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$.

Choose sequence $b_i^{(v)}$ $(1 \leqslant i \leqslant m)$ such that $b_i^{(v)} \to a_i$ $(v \to \infty)$, and for any $v \in \mathbf{N}$, $b_1^{(v)} > b_2^{(v)} > \cdots > b_m^{(v)} > 0$. Let

$$r_v(s) = \begin{pmatrix} \tanh b_1^{(v)} s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh b_2^{(v)} s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh b_m^{(v)} s & 0 & \cdots & 0 \end{pmatrix},$$

by (i), $\beta_{r_v}(\boldsymbol{X}) = \sum_{k=1}^{m} b_k^{(v)}\alpha_k$. Letting $v \to \infty$ gives $\beta_r(\boldsymbol{X}) = \sum_{k=1}^{m} a_k\alpha_k$.

Theorem 2. In $\mathfrak{R}_{RI}(m, n)$, for any geodesic ray through $\boldsymbol{O}$

$$r(s) = \boldsymbol{U}_0\begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tanh a_m s & 0 & \cdots & 0 \end{pmatrix}\boldsymbol{V}_0,$$

$$\sum_{i=1}^{m} a_i^2 = 1,\ 1 \geqslant a_1 \geqslant \cdots \geqslant a_m \geqslant 0,$$

where $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ are $m \times m$ and $n \times n$ real orthogonal matrices respectively, the Busemann function $\beta_r(\boldsymbol{X})$ determined by $r(s)$ is

$$\beta_r(\boldsymbol{X}) = -\frac{1}{2}\sum_{k=1}^{m} a_k \log d_{kk}(\boldsymbol{X}),$$

where

$$\begin{pmatrix} d_{11}(\boldsymbol{X}) & * & \cdots & * \\ * & d_{22}(\boldsymbol{X}) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_{mm}(\boldsymbol{X}) \end{pmatrix} = (\boldsymbol{U}_0 - \boldsymbol{X}\boldsymbol{V}_{01}')^{-1}(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}')(\boldsymbol{U}_0' - \boldsymbol{V}_{01}\boldsymbol{X}')^{-1},$$

and $\boldsymbol{V}_0' = (\boldsymbol{V}_{01}', \boldsymbol{V}_{02}')_{n\times n}$, $\boldsymbol{V}_{01}$ and $\boldsymbol{V}_{02}$ are $m \times n$ and $(n-m) \times n$ matrices respectively.

Proof. Denote $r(s) = \boldsymbol{U}_0 r_0(s)\boldsymbol{V}_0$.

$$\begin{aligned}\beta_r(\boldsymbol{X}) &= \lim_{s\to+\infty}(d(\boldsymbol{X}, r(s)) - s) \\ &= \lim_{s\to+\infty}(d(\boldsymbol{X}, \boldsymbol{U}_0 r_0(s)\boldsymbol{V}_0) - s) \\ &= \lim_{s\to+\infty}(d(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0', r_0(s)) - s) \\ &= \beta_{r_0}(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0').\end{aligned}$$

By Theorem 1,

$$\beta_{r_0}(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0') = -\frac{1}{2}\sum_{k=1}^{m} a_k \log d_{kk}^*(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0'),$$

where $d_{kk}^*(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0')$ satisfy

$$\begin{aligned}\boldsymbol{G}_0 := &\begin{pmatrix} d_{11}^*(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0') & * & \cdots & * \\ * & d_{22}^*(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0') & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_{mm}^*(\boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0') \end{pmatrix} \\ &= (\boldsymbol{I} - \boldsymbol{X}_1^*)^{-1}(\boldsymbol{I} - \boldsymbol{U}_0'\boldsymbol{X}\boldsymbol{V}_0'\boldsymbol{V}_0\boldsymbol{X}'\boldsymbol{U}_0)(\boldsymbol{I} - \boldsymbol{X}_1^{*\prime})^{-1}\end{aligned}$$

$$= (I - X_1^*)^{-1}(I - U_0'XX'U_0)(I - X_1^{*'})^{-1}$$
$$= (U_0 - XV_{01}')^{-1}(I - XX')(U_0' - V_{01}X')^{-1},$$

where $(X_1^{(m)*}, X_2^{(n-m)*})_{m\times n} = U_0'XV_0'$, $V_0' = (V_{01}', V_{02}')_{n\times n}$, V_{01} and V_{02} are $m\times n$ and $(n-m)\times n$ matrices respectively.

$d_{kk}(X) := d_{kk}^*(U_0'XV_0')$. Then

$$\begin{pmatrix} d_{11}(X) & * & \cdots & * \\ * & d_{22}(X) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_{mm}(X) \end{pmatrix} = (U_0 - XV_{01}')^{-1}(I - XX')(U_0' - V_{01}X')^{-1},$$

and $\beta_r(X) = -\dfrac{1}{2}\sum_{k=1}^{m} a_k \log d_{kk}(X)$.

4 The Busemann function on $\mathfrak{R}_{\mathrm{IV}}(n)$

Corollary. In $\mathfrak{R}_{RI}(2,n)$, for any geodesic ray through O

$$r(s) = \begin{pmatrix} \tanh a_1 s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \tanh a_2 s & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$
$$a_1^2 + a_2^2 = 1, \quad 1 \geqslant a_1 \geqslant a_2 \geqslant 0,$$

the Busemann function $\beta_r(X)$ determined by $r(s)$ is

$$\beta_r(X) = -\frac{1}{2}(a_1 \log d_{11}(X) + a_2 \log d_{22}(X)),$$

where $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{pmatrix}$, $X_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$.

The Cartan domain of the fourth type is the following domain[7]

$$\mathfrak{R}_{\mathrm{IV}}(n) = \{z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n : 1 + |zz'|^2 - 2|z|^2 > 0, 1 - |zz'| > 0\}.$$

The invariant metric of $\mathfrak{R}_{\mathrm{IV}}(n)$ is induced from the invariant metric of $\mathfrak{R}_{RI}(2,n)$[8] and denoted by d_{IV}. It can be checked that the real analytic transformation

$$f: \mathfrak{R}_{\mathrm{IV}}(n) \longrightarrow \mathfrak{R}_{RI}(2,n)$$

$X = \dfrac{1}{|zz'|^2 - 1}\begin{pmatrix} (\overline{zz'} - 1)z + (zz' - 1)\overline{z} \\ -i(zz' + 1)\overline{z} + i(\overline{zz'} + 1)z \end{pmatrix}$ is an isometric transformation.

Therefore, f maps the geodesic ray in $\mathfrak{R}_{\mathrm{IV}}(n)$ onto the geodesic ray in $\mathfrak{R}_{RI}(2,n)$.

Because any z_0, $z_1 \in \mathfrak{R}_{\mathrm{IV}}(n)$, there is $\varphi_{\mathrm{IV}} \in \mathrm{Aut}(\mathfrak{R}_{\mathrm{IV}}(n))$ such that $\varphi_{\mathrm{IV}}: (z_0, z_1) \longmapsto (0, \lambda)$, where $\lambda = (\lambda_1, i\lambda_2, 0, \cdots, 0)$, $\lambda_1 \geqslant \lambda_2 \geqslant 0$ and $1 > \lambda_1 + \lambda_2$. Therefore we only need to compute the Busemann function determined by the geodesic ray joining 0 to $\lambda = (\lambda_1, i\lambda_2, 0, \cdots, 0)$.

For any $z \in \mathfrak{R}_{\mathrm{IV}}(n)$,

$$Y := f(z) = \frac{1}{|zz'|^2 - 1}\begin{pmatrix} (\overline{zz'} - 1) & (zz' - 1) \\ i(\overline{zz'} + 1) & -i(zz' + 1) \end{pmatrix}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}.$$

We can check that f maps 0 and $\lambda = (\lambda_1, i\lambda_2, 0, \cdots, 0)$ to O and $\Lambda_{\mathbf{R}}$ respectively, where

$$\Lambda_{\mathbf{R}} = \begin{pmatrix} \dfrac{2\lambda_1}{1 + \lambda_1^2 - \lambda_2^2} & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{2\lambda_2}{1 - \lambda_1^2 + \lambda_2^2} & 0 & \cdots & 0 \end{pmatrix}.$$

When $\lambda_1 \geqslant \lambda_2 \geqslant 0$, $1 > \lambda_1 + \lambda_2$, it is easy to check that

$$1 > \frac{2\lambda_1}{1 + \lambda_1^2 - \lambda_2^2} \geqslant \frac{2\lambda_2}{1 - \lambda_1^2 + \lambda_2^2} \geqslant 0.$$

f also maps the geodesic ray $\omega(s)$ joining O to λ in $\mathfrak{R}_{\mathrm{IV}}(n)$ onto the geodesic ray $\omega_R(S)$ joining O to $\Lambda_{\mathbf{R}}$ in $\mathfrak{R}_{RI}(2,n)$.

By the definition of the Buseman function, we have $\beta_\omega(z) = \beta_{\omega_R}(Y)$.

Let $s_0 := d_{\mathrm{IV}}(0, \lambda) = d(O, \Lambda_{\mathbf{R}})$,

$$a_1 = \frac{1}{2s_0}\log\frac{1 + \dfrac{2\lambda_1}{1 + \lambda_1^2 - \lambda_2^2}}{1 - \dfrac{2\lambda_1}{1 + \lambda_1^2 - \lambda_2^2}} = \frac{1}{2s_0}\log\frac{(1 + \lambda_1 + \lambda_2)(1 + \lambda_1 - \lambda_2)}{(1 - \lambda_1 - \lambda_2)(1 - \lambda_1 + \lambda_2)},$$

$$a_2 = \frac{1}{2s_0}\log\frac{1 + \dfrac{2\lambda_2}{1 - \lambda_1^2 + \lambda_2^2}}{1 - \dfrac{2\lambda_2}{1 - \lambda_1^2 + \lambda_2^2}} = \frac{1}{2s_0}\log\frac{(1 + \lambda_1 + \lambda_2)(1 - \lambda_1 + \lambda_2)}{(1 - \lambda_1 - \lambda_2)(1 + \lambda_1 - \lambda_2)}.$$

Write $z = (z_1, z_2, \cdots, z_n)$, $Z_1 = (z_1, z_2)$,

$$Y_1 := f(z) = \frac{1}{|zz'|^2 - 1}\begin{pmatrix} (\overline{zz'} - 1) & (zz' - 1) \\ i(\overline{zz'} + 1) & -i(zz' + 1) \end{pmatrix}\begin{pmatrix} Z_1 \\ \overline{Z_1} \end{pmatrix}.$$

Then by Corollary, we can prove the following theorem:

Theorem 3. For $\mathfrak{R}_{\mathrm{IV}}(n)$, suppose the geodesic ray joining O to $\lambda = (\lambda_1, i\lambda_2, 0, \cdots, 0)$ is $\omega(s)$. Then the Busemann function determined by $\omega(s)$ is

$$\beta_\omega(z) = -\frac{1}{2}(a_1 \log d_1(z) + a_2 \log d_2(z)),$$

where a_1, a_2 is as above, and

$$\begin{pmatrix} d_1(z) & * \\ * & d_2(z) \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{Y}_1)^{-1}(\boldsymbol{I} - \boldsymbol{Y}\boldsymbol{Y}')(\boldsymbol{I} - \boldsymbol{Y}_1')^{-1}.$$

References

1 Cheeger, J. et al. Comparison Theorems in Riemannian Geometry. Amsterdam: North Holland Publ. Co., 1975.

2 Eberlein, P. et al. Geodesic flows on negatively curved manifolds I. Ann. of Math., 1972, 95: 492.

3 Eberlein, P. et al. Geodesic flows on negatively curved manifolds II. Trans. Amer. Math. Soc., 1973, 178: 57.

4 Zhong, J. Q. The Busemann function on classical domain, Acta Mathematica Sinica, 1990, 33: 577.

5 Lu, Q. K. The New Results of the Classical Manifolds and the Classical Domains. Shanghai: Shanghai Scientific & Technical Publishers (in Chinese), 1997.

6 Kou, M. et al. The Busemann function of the third classical domain. Advances in Mathematics, 2001, 30: 414.

7 Lu, Q. K. The Classical Manifolds and the Classical Domains, Shanghai: Shanghai Scientific & Technical Publishers (in Chinese), 1963.

8 Faraut, J. et al. Analysis and geometry on complex homogeneous domains. Progress in Mathematics, 2000, 185: 283.